The Analytical Research of Optimal Control Laws for the Flapping Wing Motion for Hovering Flight

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The air vehicles with flapping wing(s) for several types of flight tasks can be better than airplanes and helicopters. But this advantage can be realized only if the efficiency of this vehicle is higher than for other types. One of the components of total efficiency is the efficiency of a flapping wing, which is strongly dependent on the kinematics of wing motion. Considered is the problem of flapping wing motion optimization in quasi-two-dimension case for the energy consumption minimization for the fixed vertical aerodynamical force in presence of restrictions on the trajectory. Center of gravity coordinates of the vehicle are assumed to be constant (hovering flight). Pontryagin's maximum principle is used for the investigation. The comparison between the efficiencies of the flapping wing and the propeller is made.

Nomenclature

CG	=	center o gravity
C_D	=	drag force coefficient
C_{D0}	=	drag force coefficient at zero lift
C_L	=	lift force coefficient
D	=	drag force
E	=	energy consumed
F_Y	=	vertical component of the aerodynamical force
H	=	Hamilton function
L	=	lift force
P_X, P_Y	=	conjugate variables for u, v
S	=	wing area
t	=	time
T	=	final moment of time
u	=	horizontal component of wing velocity
v	=	vertical component of wing velocity
V	=	wing velocity
x,y	=	wing coordinates
W	=	power of aerodynamical force
ρ	=	air density
γ	=	wing trajectory inclination angle

I. Introduction

The generation of the lift and thrust through the flapping wing motion in some cases can be preferable in comparison with the traditional methods (propeller, jet engine etc.). However, it is rather evident that the

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efficiency of the flapping wing is strongly dependent on the kinematics of the motion and rotation of the wing.

In most articles concerning flapping wings the shape of the wing trajectory is defined a priori (one of the most popular is sine function) or defined by the kinematics of flapping mechanisms and only the parameters of this trajectory such as amplitude and phase shift are optimized. But there is no arguments proving that this type of motion is optimal. In this article the authors make the attempt to find the wing trajectory that can be optimal for the energy consumption.

Considered is the problem of the search of the rational laws of the flapping wing motion for the minimization of energy consumption at the fixed value of the vertical force produced by the wing for the zero aircraft velocity (hovering flight). The trajectory of the motion of the flapping wing is restricted by some boundaries which are determined by the construction of the wing and the motion producing mechanisms. Pontryagin's maximum principle[1] fits very well for this type of optimization problem, so it was chosen for the solution.

II. Problem statement

Considered is the problem of energy consumption minimization of the flapping wing with the fixed wing span for the hovering flight with the fixed value of the aerodynamical force vertical component. No restrictions on the horizontal component of aerodynamical force are imposed.

Let's consider this problem in the Cartesian coordinate system Oxyz attached to the Earth surface. Axis Ox is directed horizontally, axis Oy is directed vertically. Assume that the wing moves in such a way that any two points on it move in parallel planes. Assume that these planes are parallel to Oxy plane. In this case the velocity vector of the wing is always parallel to Oxy, also the wing can rotate, and the rotation axis is parallel to Oz.

Assume that the aerodynamical characteristics of the wing at any moment of time only depend on the velocity and orientation of the wing at this moment. It should be noted that all the effects of three-dimensionflow) are valid for this assumption. Also assume that the wing is a rigid body with the zero mass and zero inertia momentum.

If the horizontal and vertical velocity components are u and v, respectively, then the wing velocity is defined as

$$V = \sqrt{u^2 + v^2} \ . \qquad (1)$$

From Ref. 1, the lift force L is perpendicular to V and can be defined as

$$L = C_L \rho \frac{V^2}{2} S \,,$$

and drag force D is directed opposite to the velocity vector and is defined as

$$D = C_D \rho \frac{V^2}{2} S \,,$$

(see Fig. 1).

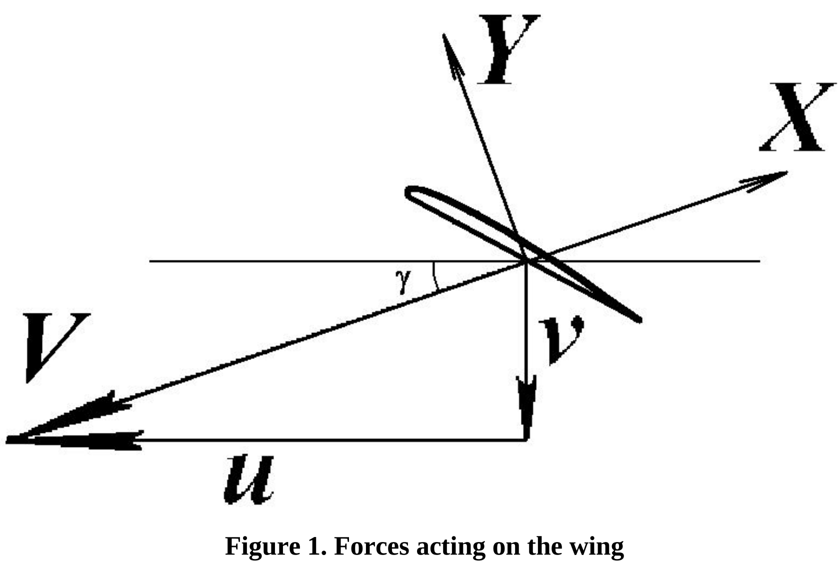

Figure 1. Forces acting on the wing

So, the vertical component of the aerodynamical force F_Y is (see fig.1)

$$F_Y = C_L \rho \frac{V^2 S}{2} \frac{u}{V} - C_D \rho \frac{V^2 S}{2} \frac{v}{V} = C_L \rho \frac{VuS}{2} - C_D \rho \frac{VvS}{2}. \tag{2}$$

The power W of the aerodynamical force is

$$W = C_D \rho \frac{V^3 S}{2},$$

and the energy consumed E is

$$E = \int_0^T C_D \rho \frac{V^3 S}{2} dt. \tag{3}$$

Assume that C_D depends only on C_L.

As the wing mass and inertia momentum are assumed to be zero, then the velocity and wing orientation can be changed instantly.

The wing motion is restricted by a closed loop, so as

$$\Phi(x,y) \le 0, \tag{4}$$

where Φ is a sertain function. It is rather evident that for high values of T the wing at some moments will move along the restriction or "touch" it.

Solved is the problem of optimal control for the wing on the basis of Pontryagin's maximum principle[2]. The variables C_L, u, v are chosen as control variables.

As there exist some limitations on the phase variables, we should take into account the kinematical relationships

$$\dot{x} = u, \dot{y} = v \tag{5}$$

There are no control variables in the boundary equation, so we should use the condition[2]

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = \frac{\partial \Phi}{\partial x} u + \frac{\partial \Phi}{\partial y} v = 0. \tag{6}$$

According to equations (1-6), Hamilton function for this problem is[2]

$$\mathbf{H} = P_X u + P_Y v + P_F \left(C_L \rho \frac{\sqrt{u^2+v^2} uS}{2} - C_D \rho \frac{\sqrt{u^2+v^2} vS}{2} - F_Y \right) + C_D \rho \frac{\left(\sqrt{u^2+v^2}\right)^3 S}{2} + \mu \left(\frac{\partial \Phi}{\partial x} u + \frac{\partial \Phi}{\partial y} v \right)$$

Where P_F - Lagrange multiplier for condition (2), $\mu(t)$ - Lagrange multiplier, $\mu>0$ on the boundary, $\mu=0$ inside the boundary.

The optimality condition for control variable C_L is[2]

$$\frac{\partial \mathrm{H}}{\partial C_L} = 0, \Rightarrow P_F u + \left(V^2 - P_F v\right) \frac{\mathrm{d}C_D}{\mathrm{d}C_L} = 0. \tag{7}$$

Inside the boundaries optimality conditions for u and v are[2] (as $\mu=0$)

$$\frac{\partial \mathbf{H}}{\partial u} = 0, \Rightarrow P_X + P_F C_L \frac{\rho S}{2\sqrt{u^2+v^2}} \left(2u^2+v^2\right) - C_D \frac{\rho u S}{2} \left(P_F \frac{v}{\sqrt{u^2+v^2}} - 3\sqrt{u^2+v^2} \right) = 0, \tag{8}$$

$$\frac{\partial \mathbf{H}}{\partial v} = 0, \Rightarrow P_Y + P_F C_L \frac{\rho uvS}{2\sqrt{u^2+v^2}} - C_D \frac{\rho S}{2} \left(P_F \frac{\left(2v^2+u^2\right)}{\sqrt{u^2+v^2}} - 3v\sqrt{u^2+v^2} \right) = 0. \tag{9}$$

As the Hamilton function inside the boundaries does not depend explicitly on the coordinates x and y, so the conjugate variables P_x and P_y are constants inside the boundaries[2]. So, there are four algebraic equations (2), (7-9) for four variables P_F, C_L, u, v. The solution of this system of equation is one or more sets of fixed values of P_F, C_L, u, v. This means that inside the boundaries these variables are constants, so the wing inside the boundaries should be moved along the straight line with constant velocity and constant values of aerodynamical forces.

On the boundary conditions (6) and (8) and also the optimality conditions for u and v should be valid:

$$\frac{\partial \mathbf{H}}{\partial u} = 0, \Rightarrow P_X + \mu \frac{\partial \Phi}{\partial x} + P_F C_y \frac{\rho S}{2\sqrt{u^2+v^2}} \left(2u^2+v^2\right) - C_x \frac{\rho u S}{2} \left(\frac{P_F v}{\sqrt{u^2+v^2}} - 3\sqrt{u^2+v^2} \right) = 0, \tag{10}$$

$$\frac{\partial \mathbf{H}}{\partial v} = 0, \Rightarrow P_Y + \mu \frac{\partial \Phi}{\partial y} + P_F C_y \frac{\rho u v S}{2\sqrt{u^2+v^2}} - C_x \frac{\rho S}{2}\left(P_F \frac{\left(2v^2+u^2\right)}{\sqrt{u^2+v^2}} - 3v\sqrt{u^2+v^2} \right) = 0 . \quad (11)$$

The conditions for conjugate variables on the boundary are

$$\begin{cases} \dot{P}_X = -\dfrac{\partial \mathbf{H}}{\partial x} = -\mu\left(\dfrac{\partial^2 \Phi}{\partial x^2} u + \dfrac{\partial^2 \Phi}{\partial x \partial y} v \right) \\ \dot{P}_Y = -\dfrac{\partial \mathbf{H}}{\partial y} = -\mu\left(\dfrac{\partial^2 \Phi}{\partial x \partial y} u + \dfrac{\partial^2 \Phi}{\partial y^2} v \right) \end{cases} \quad (12)$$

So, at any point on the boundary system of equations (2), (6), (8), (10), (11) the P_F, Cy, u, v and μ can be obtained as functions of coordinates and conjugate variables. With the help of (12) one can obtain the wing coordinates as a function of time.

It should be noted that the function Φ can be rather complex. Moreover, even for "simple" shape of this function the system (2), (6), (8), (10)-(12) is rather complex itself. And, at last, one should "connect" the solutions inside the boundaries and on the boundary. These reasons give practically no chances to obtain the analytical solution in the general case.

But for the set of rather simple but characteristical cases the analytical solution can be obtained. They will be considered below.

1. There are no limitations imposed on the trajectory. The coordinates at the final moment are not fixed. (This case can be realized if the total time of the process is rather short and the boundaries are far enough).

In this case the conjugate variables P_X and P_Y are equal to zero for all the process time. Then equations (8) and (9) are as following:

$$P_F C_y \left(2u^2+v^2\right) - P_F C_x uv + 3C_x uV^2 = 0 ,$$

$$P_F C_y uv - P_F C_x \left(2v^2+u^2\right) + 3C_x V^2 v = 0 ,$$

That gives the condition

$$\frac{v}{u} = tg\gamma = -\frac{C_D}{C_L} .$$

Assuming that the dependence for C_D is given by the equation[1]

$$C_D = C_{D0} + AC_L^2, \quad (13)$$

from (7)-(9) one can obtain following relationships:

$$C_L = \frac{\sqrt{1-4C_{D0}-\sqrt{1-32AC_{D0}}}}{2A}, \frac{v}{u} = -\frac{1-\sqrt{1-32AC_{D0}}}{2\sqrt{1-4C_{D0}-\sqrt{1-32AC_{D0}}}} .$$

It should be noted that for $C_L>0$ $\gamma<0$, so the sing moves slightly down.

2. There exists boundary $y \geq y_{\min}$, and at the initial moment $y=y_{\min}$. The coordinates at the final moment are not fixed.

In this case P_X is equal to zero for all the process time. Assume that the wing at the final moment is not on the boundary. P_Y should be equal to zero[2]. From the above results, it means that the wing is to move slightly down at this moment. But we know that inside the boundaries the wing cannot change the direction of the motion, so it should move all the time towards the lower boundary. But it cannot move in such a way because at the first moment it was on the boundary. So, the only variant of motion in this case is the motion along the horizontal boundary.

In this case equations (7), (10), (13) give the condition

$$3C_{D0} = AC_L^2.$$

3. There exists the boundary $y \geq y_{\min}$, and at the initial moment $y>y_{\min}$. The coordinates at the final moment are not fixed.

From the previous results it is evident that the wing motion should finish on the boundary. The general case of this trajectory is shown in Fig. 2. The only problem is to find the value of angle γ which provides the minimum of energy consumption.

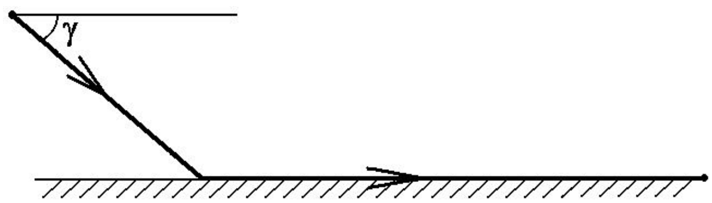

Figure 2. Possible trajectory of wing for case 3.

Direct analytical investigations show that the value of this angle should be such that the wing would be on the boundary only at the final moment. The trajectory corresponding to this result is shown in Fig. 3.

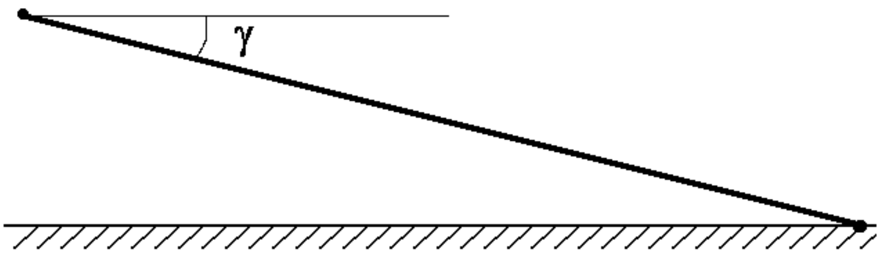

Figure 3. Optimal wing trajectory for case 3.

4. The wing can move inside the restrictions $y_{\max} \geq y \geq y_{\min}$. For this case one can note that, first of all, the trajectory obtained in the previous chapter can be valid. Second, the extra limitation can not improve the result. So the value of the energy consumption in this chapter cannot be lower than in the previous chapter. As the solution from the previous chapter is valid, so it is one of the optimal solutions with the same result or the only one solution.

5. The wing can move inside the boundaries $y_{\max} \geq y \geq y_{\min}$, $x_{\max} \geq x \geq x_{\min}$. If we assume that the optimal trajectory in this case is a broken line with the same absolute values of trajectory inclination angles and with the final point on the lower boundary (see Fig. 4), then the value of the energy consumed will be equal to the result of previous chapter for the same upper and lower limits.

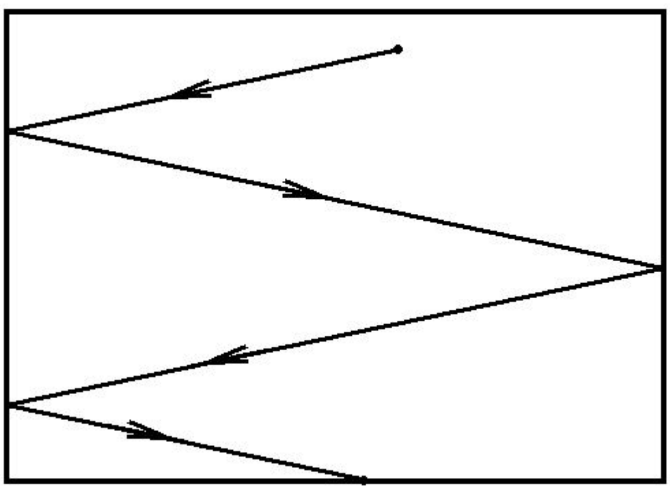

Figure 4. Optimal wing trajectory for case 5.

It should also be noted that for the case of $T \to \infty$ the angle γ tends to zero, so for this case in the first approximation one can assume that the wing is to move practically horizontally.

III. Results discussion

Now it is necessary to discuss some peculiarities of the solution obtained.

First of all, in should be mentioned that the wing profile usually has sharp trailing edge and thick leading edge (it is necessary for the good flow pattern). So, in case N5 mentioned above the leading edge of the wing is in front during the motion in "positive" direction. But during the motion in negative direction the trailing edge can be in front. This can lead to flow separation on the wing. To prevent this problem one can rotate the wing in such a way that the leading edge will always be in front as shown in Fig. 5. By the way, the hummingbird during the hovering move its wings in such a way (see Fig. 6)[3].

Next, in the fifth case one cannot define the "preferable" horizontal direction. So, the mean value of thehorizontal aerodynamical force is equal to zero. So, at average, the aircraft center of gravity will not move in the horizontal direction. Also, if the mean position of CG is in the middle between the vertical restrictions, then the mean torque from aerodynamical forces will be zero.

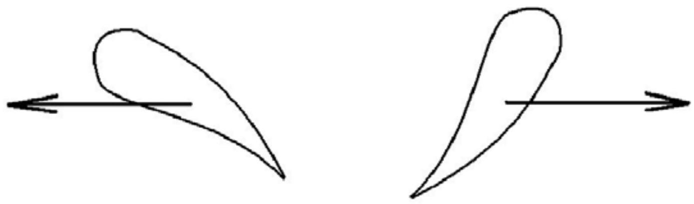

Figure 5. Wing motion during the flapping.

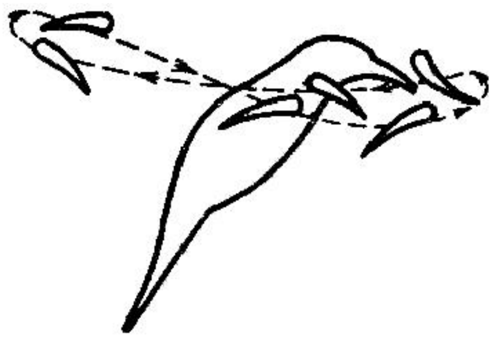

Figure 6. Hummingbird's wing motion during the hovering[4].

And, at least, it is necessary to compare the flapping wing and helicopter efficiencies in the statement of this problem.

As it was mentioned above, in the fifth case for the long periods of time one can assume that the flapping wing moves horizontally, and the velocity and aerodynamical forces are the same for the "positive" and "negative" directions of motion. If we do not take into account the effects of non-stationarity, then nothing changes if we will move the wing with the same velocity and with the same angle of attack in one direction ("positive" or "negative"). But this case is equal to the motion of the quasi-two-dimensional motion of a helicopter blade.

One can imagine this comparison with the following illustration. Let the flapping wing be placed on the one end of a long bulk, and the other end of this bulk always has zero velocity. If the length of this bulk is high enough in comparison with the wing span of a flapping wing, then the motion of the flapping wing will be practically the same as in the problem considered. The solution obtained in the fifth case is equal to that of the bulk is moved (rotates) in the horizontal plane with constant velocity in positive and negative direction . But the vertical force and the power will be the same in both directions. So, if we move this wing only in one direction all the time, the result will be the same.

This means that the efficiency of the flapping wing in the hovering flight is not higher that the efficiency of the helicopter with the same wing (blade).

It should be noted that in Ref. 4 the authors have come nearly to the same conclusion for the comparison of three-dimensional flapping wing and helicopter.

Conclusions

1. Obtained were the optimal trajectories of the wing motion for the minimization of the energy consumption at the fixed value of the vertical component of the aerodynamical force in the cases of presence and absence of the limitations on the wing trajectory.

2. Inside the boundaries the wing should move along the straight line with constant values of the velocity and C_L.

3. For the limitations of the rectangle, the trajectory should be a broken line with the constant absolute value of inclination angle, the changes of the trajectory directions must be on the vertical restrictions. The value of the inclination angle can be found from the condition that the terminal point of the trajectory should be on the lower horizontal restriction.

4. In the case of the rectangle limitations the mean value of of horizontal aerodynamical forces and torques is equal to zero.

5. For this problem statement the efficiency of the flapping wing is not higher than the efficiency of the helicopter.

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